Knots, Genus, and Surfaces

Nairi Davidian; Winnie Gong

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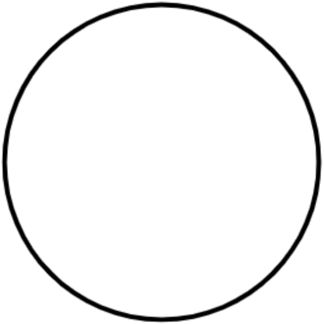
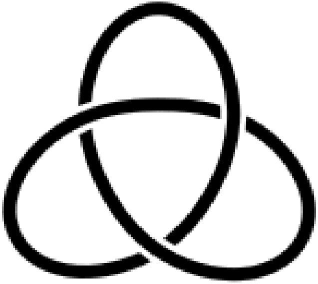
Introduction to Knots

Formal definition of a knot: A closed curve embedded in three-dimensional space. Examples of mathematical knots include the unknot and trefoil knot:



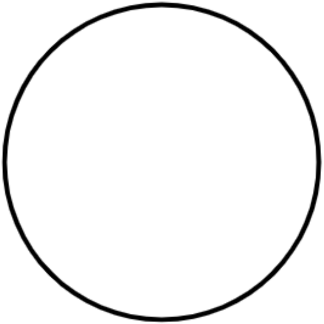
(a) Unknot



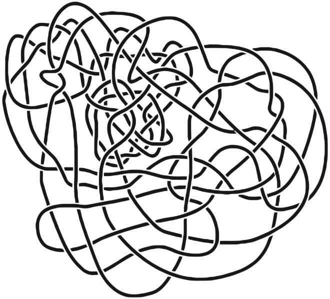
(b) Trefoil Knot

Knot Equivalence

Equivalent knots: A pair of knots such that one can be continuously deformed into the other without cutting or passing through itself



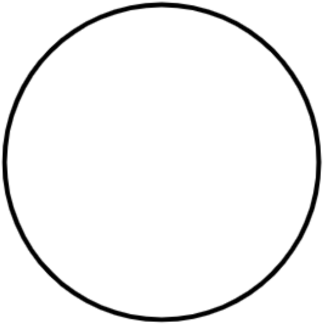
(a) Unknot



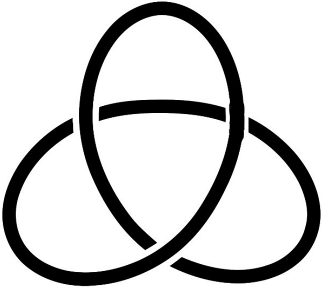
(b) Haken Gordian Knot

Invariants

Invariant: A mathematical quantity or property that is not changed under continuous deformations such as stretching, bending, and twisting; they help us tell knots apart



(a) Unknot



(b) Right-handed Trefoil Knot

Surfaces

Surface: A manifold that looks like a plane locally, like the glaze of a DONUT!!! There are two main types of surfaces: surfaces without boundary and surfaces with boundary. Boundaries are like bites you take out of a donut.

(a) Donuts With Glaze

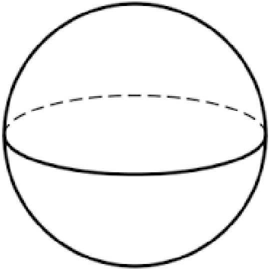
(b) Donuts with Bites

Genus

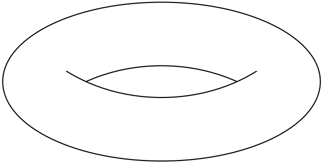
Genus: a non-negative integer equivalent to the maximum number of closed curves that can be cut out of the surface without disconnecting the manifold.

Surface Without Boundary

Surfaces without boundary include a sphere and torus. The genus of a surface without boundary is the number of holes on a surface without boundary.



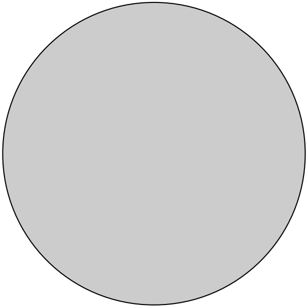
(a) Sphere Genus 0



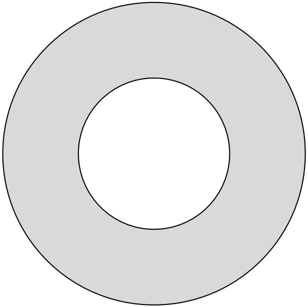
(b) Torus Genus 1

Surface With Boundary

Surfaces with boundary include the disk and annulus. The genus of a surface with boundary can be found by capping it off.



(a) Disk Genus 0



(b) Annulus Genus 0

Euler Characteristic

Triangulation: V vertices, E edges, and F faces

$$X = V - E + F$$

Proposition: Different triangulations of the same surface yield the same characteristic.

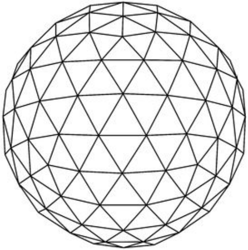
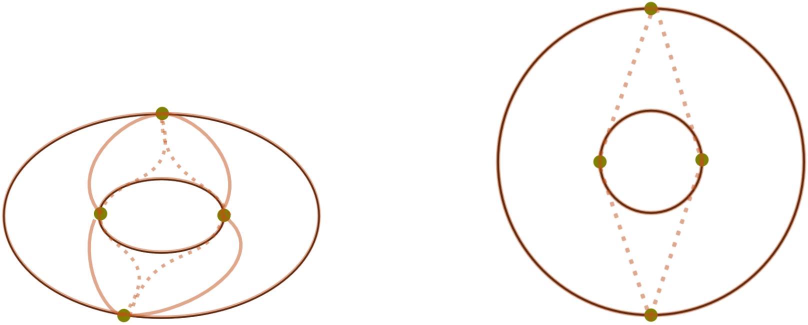


Figure: Triangulation of a Sphere

A triangulation of the torus, which has genus 1, shown below has 4 vertices, 12 edges, and 8 faces:



(a) Triangulation of Torus (b) Top View

$$X(T) = 4 - 12 + 8 = 0$$

Connected Sum: Formula for the Euler Characteristic of the connected sum of any two surfaces without boundary.

$$X(S) = X(T_1) + X(T_2) - 2.$$

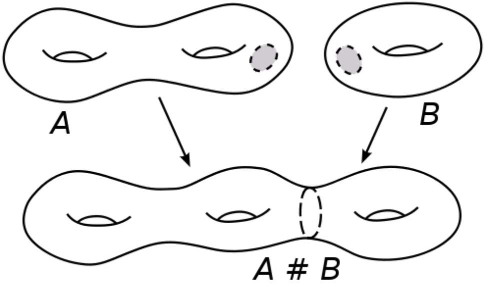
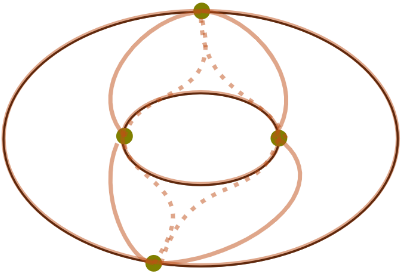


Figure: Connected Sum

Genus and Euler Characterstic

Proposition: The Euler characteristic X of a surface without boundary of genus g is $X(\sigma) = 2 - 2g(\sigma)$ for all positive integers.

Base Case: A Torus



$$V = 4 \quad E = 12 \quad F = 8$$

$$X = V - E + F = 4 - 12 + 8 = 0$$

$$g = 1$$

$$X = 2 - 2 = 2 - 2(1) = 0$$

The base case holds.

Induction: Assume that $X(\sigma)$ holds for a surface of genus $g = k$, $X(S_k) = 2 - 2k$. Consider a surface, $S_{(k+1)}$, of genus $g = k + 1$:

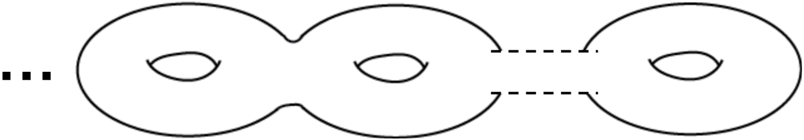


Figure: $g = k + 1$

$$X(S_{(k+1)}) = X(S_k) + X(T) - 2$$

$$= 2 - 2k + 0 - 2$$

$$= 2 - 2k - 2$$

$$= 2 - 2(k + 1)$$

$$\Rightarrow X(S_{(k+1)}) = 2 - 2(k + 1)$$

Conclusion: By mathematical induction, the formula $X(\sigma) = 2 - 2g$ is true for all surfaces without boundary, where g is the genus and X is the Euler characteristic.

A similar formula for surfaces with boundaries can be derived from $X(\sigma) = 2 - 2g$:

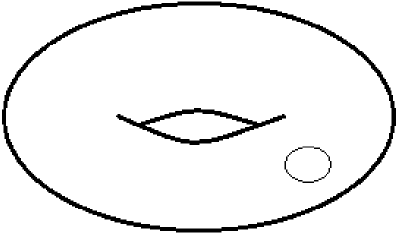
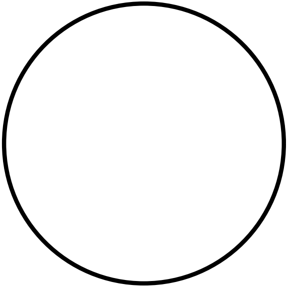


Figure: Torus with Boundary

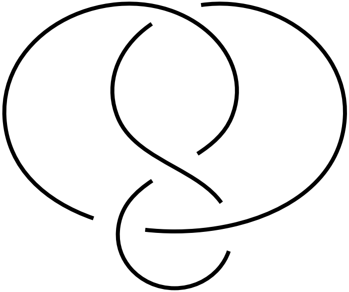
$$X(\sigma) = 2 - 2g - b$$

Seifert Surface

A Seifert Surface: special type of surface with one boundary component associated with a knot or link in three-dimensional space. The genus of a knot is the least genus of any Seifert surface for that knot:

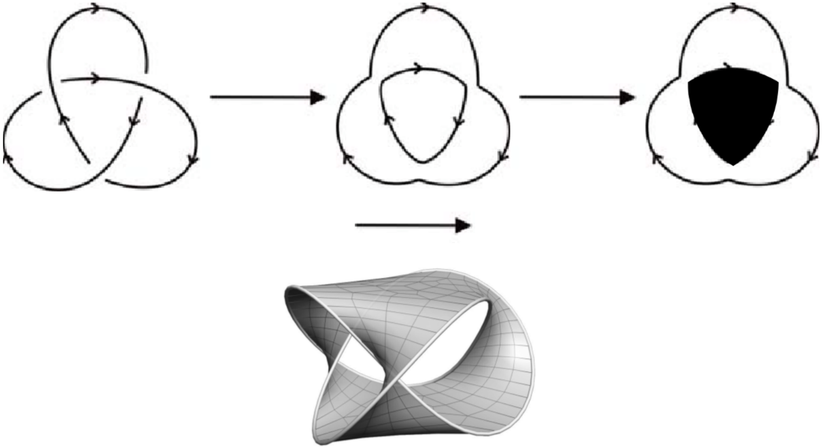


(a) Unknot Genus 0



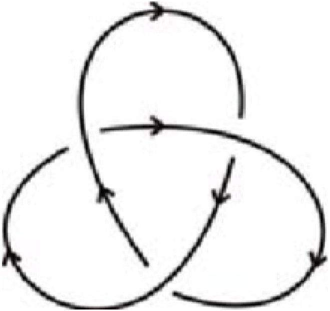
(b) Figure Eight Knot Genus 1

Turning Knots into Seifert Surfaces

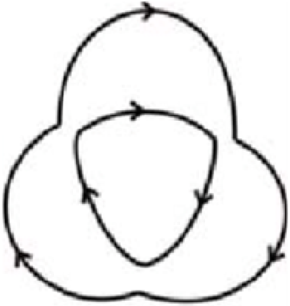


Euler Characteristic and Genus a Seifert Surface

When s is the number of Seifert circles and c is the number of crossings of a corresponding knot, its Euler characteristic is $X = s - c$.



(a) Trefoil Knot Crossings



(b) Trefoil Knot Seifert Circles

$$X = 2 - 3 = -1$$

$$X(g) = 2 - 2g - b$$

$$X(g) = 2 - 2g - 1 = 1 - 2g$$

$$X = s - c$$

$$\Rightarrow 1 - 2g = s - c$$

$$2g = 1 - s + c$$

$$g = \frac{1 - s + c}{2}$$

This formula can be utilized to deduce the properties of groups of knots

Twist Knots

Twist knots are a specific class of knots that include the trefoil and figure eight knots:

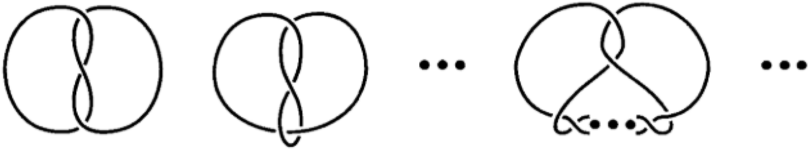
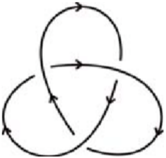


Figure: Twist Knots

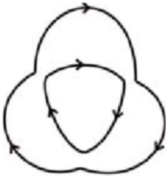
A property of twist knots is for a twist knot with c crossings and s Seifert circles, $s = c - 1$.

Proof for $s = c - 1$ for Twist Knots

Proposition: For all twist knots, $s = c - 1$.
Base Case: Trefoil Knot



(a) Trefoil Knot Crossings



(b) Trefoil Knot Seifert Circles

$$c = 3$$
$$s = 2$$
$$2 = 3 - 1$$
$$\Rightarrow s = c - 1$$

The base case holds.

Induction: Assume that the statement, $s = c - 1$, holds for $c = k$, $s = k - 1$. Consider a twist knot with $k + 1$ crossings that is obtained by adding a crossing.

Every time a crossing is added to a twist knot, the number of Seifert circles increases by 1.:

$$s = (k - 1) + 1$$

$$\Rightarrow s = (k + 1) - 1$$

Conclusion: By mathematical induction, the property $s = c - 1$ is true for all twist knots where c is the number of crossings and s is the number of Seifert circles.

Genus of Twist Knots

Equation 1:

$$s = c - 1$$

Equation 2:

$$g = \frac{1 - s + c}{2}$$

Equation 3:

$$g = \frac{1 - (c - 1) + c}{2}$$

$$g = \frac{1 - c + 1 + c}{2}$$

$$g = \frac{2}{2}$$

$$g = 1$$

All twist knots have genus 1.

Prime Knots

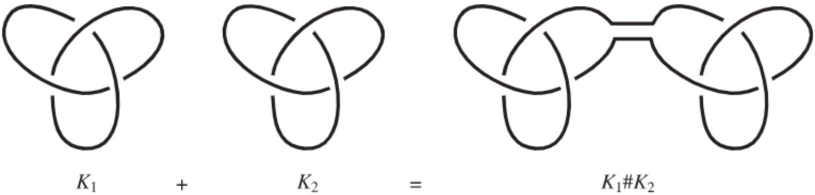


Figure: Knot Sum

Prime knots are non-trivial knots that cannot be expressed as the knot sum of two non-trivial knots (K_1 and K_2 are nontrivial knots):

$$K \neq K_1 \# K_2$$

Composite knots are knots that can be expressed as the knot sum of two more nontrivial knots:

$$K = K_1 \# K_2$$

Unknot is neither a prime knot nor a composite knot.

Genus and Prime Knots

Proposition: a knot with genus 1 must be a prime knot The

Additivity of Knot Genus theorem:

$$g(K) = g(K_1 \# K_2) = g(K_1) + g(K_2)$$

Assumption: Let K be a genus 1 knot:

$$g(K) = 1$$

$$\Rightarrow g(K_1 \# K_2) = 1$$

$$\Rightarrow g(K_1) + g(K_2) = 1$$

Case 1:

$$g(K_1) = 0$$

$$g(K_2) = 1$$

Case 2:

$$g(K_1) = 1$$

$$g(K_2) = 0$$

At least one knot must have a genus of 0. This implies one of the knots, K_1 or K_2, must be the unknot. However, K_1 and K_2 must be a non-trivial knot and the unknot is not a non-trivial knot.

Conclusion: A knot with genus 1 cannot be decomposed into two nontrivial knots. Hence, a knot with genus 1 must be a prime knot.

Twist Knots and Prime Knots

Statement 1: All twist knots have genus 1
Statement 2: All genus 1 knots are prime knots
Statement 3: All twist knots are prime knots

Acknowledgements

We would like to thank:

Our mentor Yuyuan Luo

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Our audience

Questions